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ALGEBRAS OF FUNCTIONS ON SEMITOPOLOGICAL LEFT-GROUPS

BY

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ABSTRACT. We consider various algebras of functions on a semitopological left-group $S = X \times G$, the direct product of a left-zero semigroup X and a group G. In §1 we examine various analogues to the theorem of Eberlein that a weakly almost periodic function on a locally compact abelian group is uniformly continuous. Several appealing conjectures are shown by example to be false. In the second section we look at compactifications of products $S \times T$ of semitopological semigroups with right identity and left identity, respectively. We show that the almost periodic compactification of the product is the product of the almost periodic compactifications, thus generalizing a result of deLeeuw and Glicksberg. The weakly almost periodic compactification of the product is not the product of the weakly almost periodic compactifications except in restrictive circumstances; for instance, when T is a compact group. Finally, as an application, we define and study analytic weakly almost periodic functions and derive the theorem, analogous to a classical theorem about almost periodic functions, that an analytic function which is weakly almost periodic on a single line is analytic weakly almost periodic on a whole strip.

Introduction. Semigroups of various types are the natural objects on which to study spaces of functions which are "almost periodic" in some sense or "uniformly continuous" in some sense. Much of the study of such function spaces has either concentrated on general theory with no algebraic restrictions being placed on the semigroups, or it has dealt with the special case in which the semigroup is, in fact, a group. We propose to look at a special type of semigroup, a left-group, which, although it is not a group, is yet rich enough in algebraic structure to allow a detailed analysis. A *semitopological semigroup* is a set S together with an associative binary operation ("multiplication")

$$(s, t) \longrightarrow st: S \times S \longrightarrow S$$

and a Hausdorff topology with respect to which the binary operation is separately continuous. If the multiplication is jointly continuous, then S is called a *topological semigroup*. A semigroup is called a *left-group* if it is left simple

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and right cancellative. A left-group always decomposes algebraically into a direct product $X \times G$ of a *left-zero* ($x_1x_2 = x_1$) semigroup X and a group G. For a left-group which is also a locally compact semitopological semigroup, the direct product decomposition $X \times G$ is also topological, with X a locally compact left-zero semigroup and G a locally compact topological group [3, II.2.3]. However, the decomposition need not be topological. If the left-group $S = [0, 1] \times R$, for example, is given the Knight-Moran-Pym topology of separate continuity [11], which makes S a semitopological semigroup, then the direct product decomposition is not topological. To avoid this pathology here, we define a *semitopological left-group* to be a direct product $S = X \times G$ of a left-zero topological semigroup X and a semitopological semigroup G which is algebraically a group. [Note that, if G is locally compact, then it must be a topological group and S must be a topological semigroup.]

Let S be a semitopological semigroup. Denote the C^*-algebra of all bounded, continuous complex-valued functions on S by $C(S)$. Given $s \in S$, define the *left translation operator* L_s [respectively, the *right translation operator* R_s] on $C(S)$ by

$$L_sf(t) = f(st), \quad t \in S, \qquad [R_sf(t) = f(ts), t \in S]$$

for each $f \in C(S)$. Let

$$O_L(f) = \{L_sf | s \in S\}, \qquad [O_R(f) = \{R_sf | s \in S\}].$$

A function $f \in C(S)$ is called *almost periodic* if $O_R(f)$ is relatively compact in the norm topology of $C(S)$; equivalently, f is almost periodic if $O_L(f)$ is relatively norm compact in $C(S)$. Call $f \in C(S)$ *weakly almost periodic* if $O_R(f)$ is relatively compact in the weak topology of $C(S)$; equivalently, f is weakly almost periodic if $O_L(f)$ is relatively weakly compact in $C(S)$. Denote the set of almost periodic functions on S by $AP(S)$ and the set of weakly almost periodic functions by $WAP(S)$. Both $AP(S)$ and $WAP(S)$ are C^*-subalgebras of $C(S)$. Other C^*-subalgebras of $C(S)$ are defined as follows:

(1) A function $f \in C(S)$ is in $LUC(S)$ [respectively, $RUC(S)$] if the function

$$s \longrightarrow L_sf: S \longrightarrow C(S), \qquad [s \longrightarrow R_sf: S \longrightarrow C(S)]$$

is continuous when $C(S)$ has its norm topology.

(2) A function $f \in C(S)$ is in $LMC(S)$ [respectively, $RMC(S)$] if $O_R(f)$ [respectively, $O_L(f)$] is relatively compact in the pointwise topology of $C(S)$. (The authors are preparing a comprehensive study of these last subspaces and related topics.)

The relationships among the various C^*-algebras defined above are given by the following diagram (all arrows are inclusions):

(A)
$$\begin{array}{ccccc} RUC & \longrightarrow & RMC & & \\ \uparrow & & \uparrow & \searrow & \\ AP & \longrightarrow & WAP & \longrightarrow & C(S) \\ \downarrow & & \downarrow & \nearrow & \\ LUC & \longrightarrow & LMC & & \end{array}$$

Comments about the notation LMC and the inclusion $LUC \subset LMC$ are in order. (Analogous comments can be made about RMC and RUC.) Let βS be the spectrum of $C(S)$, which is just the Stone-Čech compactification of S if S is completely regular. Then LMC turns out [12, Theorem 3.1] to be equal to the left multiplicatively continuous subspace of Mitchell [13],

$$\{f \in C(S) \mid \text{the function } s \longrightarrow \mu(L_s f) \text{ is continuous for each } \mu \in \beta S\},$$

which obviously contains LUC.

In [12] an example is given of a locally compact semitopological semigroup S for which all the inclusions of (A) are proper. In addition, neither $WAP(S) \subseteq LUC(S)$ nor $LUC(S) \subseteq WAP(S)$. On the other hand, if G is a locally compact group, then

(B)
$$\begin{array}{ccccc} & & RUC = RMC & & \\ & \nearrow & & \searrow & \\ AP \longrightarrow WAP & & & & C(G) \\ & \searrow & & \nearrow & \\ & & LUC = LMC & & \end{array}$$

Moreover, $LUC(G)$ [respectively, $RUC(G)$] is the space of all bounded complex-valued functions on G which are uniformly continuous with respect to the right [respectively, left] uniform structure on G [4, Chapter III, §3].

A left-group is, in many ways, the simplest of nontrivial semigroups. (From a semigroup viewpoint, a group is "trivial".) One is naturally led, therefore, to considerations of left-groups when moving from statements about groups to statements about semigroups. Our interest in left-groups, however, does not stem solely from the desire to look at objects which are almost groups. Rather, the initial motive for studying these functions came from the observation that the *bounded almost periodic functions depending uniformly on parameters* of Corduneanu [5, II.1] are precisely almost periodic functions on topological left-groups $S = X \times G$, where X is a subset of complex n-space C^n, and G is the additive group of real numbers, provided that the parameter space X is compact [5, Theorem 2.6, p. 54].

Several of the theorems of Corduneanu reduce to the statement that the set of almost periodic functions on $S = X \times G$ is a C^*-subalgebra of $C(S)$. Corduneanu also discussed the uniform continuity of almost periodic functions depending uniformly on parameters. In §1, we consider various analogues to Corduneanu's theorem. Besides $LUC(S)$ and $RUC(S)$, we also deal with actual uniform continuity: If $S = X \times G$ is a semitopological left-group with X a uniform space and G a topological group, let $U_L = U_L(S)$ be the space of all functions $f \in C(S)$ which are uniformly continuous when G is given the uniformity with basis

$$\{(s, t) \in G \times G | st^{-1} \in V\}$$

as V runs through the neighbourhoods of the identity in G, and S is given the product uniformity. Define $U_R = U_R(S)$ similarly using the left uniform structure on G. The spaces U_L and U_R are particularly natural to study when X is compact and hence has a unique uniformity.

Corduneanu observed that Bohr's definition of analytic almost periodic functions may be phrased in terms of almost periodic functions depending uniformly on parameters. Consequently, the definition may be phrased in terms of almost periodic functions on left-groups; and we are led naturally to various spaces of analytic functions. The only one of interest, it turns out, besides the analytic almost periodic functions, is the space of analytic weakly almost periodic functions, which we study in §3.

Each section contains a summary of the results of that section.

1. **Inclusion relationships: General theorems and counterexamples.** In [12 p. 502, Example] it was shown that the orderly sequence of inclusions and equalities that holds among the subspaces under consideration for locally compact groups [see (B) in the Introduction] breaks down to a large extent for semitopological semigroups; the semigroup of the example in [12] is explicitly assumed *not* to be topological. We show in this section that much of this breakdown can occur for topological semigroups, even for ones of very simple type, the locally compact topological left-groups. Some positive results are proved first. We remark that, in the following theorem, the hypothesis that X is a uniform space is required only for (i), (ii) and one assertion in (iv).

THEOREM 1.1. *Let $S = X \times G$ be a topological left-group such that X is a uniform space with uniformity $\mathfrak{U}$.*

(i) *$LUC \supset U_L$ and $LUC = U_L$ if X is compact.*

(ii) *$RUC \supset U_R$ and the containment can be proper even if X is compact.*

(iii) *$f \in RUC$ if and only if $f \in C(S)$ and the set of functions*

$$A_f = \{F_x | x \in X\} \subset C(G),$$

where

$$F_x(s) = f(x, s) \quad \textit{for all } s \in G,$$

is left uniformly equicontinuous.

(iv) *If* $\psi: S \longrightarrow S$ *is defined by* $\psi(x, s) = (x, s^{-1})$, *then the adjoint* ψ^* *effects an isometric isomorphism of* U_L *onto* U_R, *of WAP onto itself, and of AP onto itself. Also,* ψ^* *injects LUC into RUC if X is compact and* $RUC\backslash\psi^*(LUC)$ *can be nonvoid even if X is compact.*

PROOF. (i) It follows immediately from definitions that $LUC \supset U_L$. Suppose X is compact and $f \in LUC$. We must show $f \in U_L$, i.e., given $\epsilon > 0$, we must find $U \in \mathfrak{U}$ and a neighbourhood V of the identity e of G such that

$$|f(x, u) - f(y, t)| < \epsilon$$

whenever $(x, y) \in U$, $ut^{-1} \in V$. We do this as follows. Given any $(x, s) \in S$, we can find a neighbourhood $U_x \times V_s$ of (x, s), where U_x and V_s are neighbourhoods of x and s, respectively, such that

$$|f(x, sr) - f(y, tr)| < \epsilon/2$$

for all $r \in G$ whenever $(y, t) \in U_x \times V_s$. Thus

$$|f(x, u) - f(y, t)| < \epsilon/2$$

whenever $y \in U_x$ and $ut^{-1} \in sV_s^{-1}$. Since X is compact,

$$X = \bigcup_1^n U_{x_i}$$

for a suitably chosen x_i, $i = 1, 2, \ldots, n$. Also, there is a $U \in \mathfrak{U}$ such that, for each $y \in X$,

$$U(y) = \{y' \mid (y, y') \in U\} \subset U_{x_i}$$

for some i. Then, if $(x, y) \in U$ and $ut^{-1} \in V = \bigcap_1^n s_i V_{s_i}^{-1}$,

$$|f(x, u) - f(y, t)| \leqslant |f(x, u) - f(x_i, t)| + |f(x_i, t) - f(y, t)| < \epsilon$$

for at least one i as required. [See the remark following the proof of this theorem for a left-group $S = X \times G$ with X not compact and $LUC\backslash U_L \neq \varnothing$.]

(ii) The fact that $RUC \supset U_R$ also follows immediately from definitions. A function in $RUC\backslash U_R$ is constructed on the left-group $[0, 1] \times R$ following Theorem 1.4 ahead.

(iii) $f \in RUC$ if and only if f is continuous and

$$\lim_\alpha \sup\{\, |f(x, tr_\alpha) - f(x, tr)| \mid (x, t) \in S\} = 0$$

whenever $\{r_\alpha\}$ converges to $r \in G$. The desired conclusion follows directly from this assertion and the fact that, for a topological group G, $RUC(G)$ is just the set of functions in $C(G)$ that are uniformly continuous with respect to the left uniformity of G.

(iv) Most of the assertions here have easy proofs which we omit. That ψ^* injects LUC into RUC when X is compact follows from (i) and (ii) and the fact that ψ^* maps U_L onto U_R. [That $\psi^*(LUC)\backslash RUC$ can be nonvoid if X is not compact is shown in the next remark.] A function in $RUC\backslash\psi^*(LUC)$ is constructed on the left-group $[0, 1] \times R$ after Theorem 1.4.

REMARK. That $LUC\backslash U_L$ can be nonvoid if X is not compact follows easily from the observation that $C(X) = LUC(X)$ for every topological left-zero semigroup X. Thus, for the left-group $R \times [0, 1)$, where R, the set of real numbers, and $[0, 1) = R/Z$ have their usual metric uniformities, the function f defined by

$$f(x, s) = \sin(x^2) \quad \text{for all } x \in R,\ s \in [0, 1)$$

is in $LUC\backslash U_L$. A function in $LUC\backslash RUC$ on this same left-group is defined by

$$f(x, s) = s(1 - s) \sin xs \quad \text{for all } x \in R,\ s \in [0, 1).$$

Obviously, $f \in LUC = \psi^*(LUC)$ [notation as in Theorem 1.1, (iv)]; and $f \notin RUC$ by Theorem 1.1, (iii).

It is known that $AP \subset LUC \cap RUC$ always and that

(α) $AP(S) = RUC(S)$, and

(β) $AP(S) = LUC(S)$

at least if S is a compact semitopological semigroup [12, Lemma 4.11] or a topological left-zero semigroup. In view of the Ascoli-Arzelà theorem we have the following corollary to Theorem 1.1, (iii), which gives another noncompact setting in which the equality (α) holds. [In fact, the statements of Theorem 1.1, (iii), and the present corollary make sense and remain true for semigroups of the form $S = X \times T$ where X is a topological left-zero semigroup and T is a compact semitopological semigroup.] Together with the second example of the immediately preceding remark, this corollary also shows that the equality (β) can fail to hold in this setting.

COROLLARY 1.2. *Let $S = X \times G$ be a topological left-group with G compact. Then $AP = RUC$.*

Since bounded almost periodic functions depending uniformly on parameters are almost periodic functions on certain left-groups [see the Introduction], the next theorem can be viewed as a considerable generalization of Theorem 2.2, p. 52, of [5].

THEOREM 1.3. *Let $X \times G$ be a topological left-group with X compact. Then $AP \subset U_L \cap U_R$.*

PROOF. Since $AP \subset LUC$ always and $LUC = U_L$ in this setting we need only show $AP \subset U_R$. Suppose $f \in AP$ and $\epsilon > 0$ are given. Then $F: X \longrightarrow C(G)$, defined by

$$(F(x))(s) = f(x, s)$$

for all $x \in X$, $s \in G$, is uniformly continuous, hence there is a $U_1 \in \mathfrak{U}$, the unique uniformity of X, such that

$$\|F(x_1) - F(x_2)\| < \epsilon/3$$

whenever $(x_1, x_2) \in U_1$. Also, there is a $U \in \mathfrak{U}$ such that $U^2 \subset U_1$ and there is a finite subset $\{x_1, x_2, \ldots, x_n\} \subset X$ such that

$$\bigcup_1^n \{y \in X | (x_i, y) \in U\} = X.$$

Let V be a neighbourhood of $e \in G$ such that

$$|f(x_i, s) - f(x_i, t)| < \epsilon/3, \qquad i = 1, 2, \ldots, n,$$

whenever $s^{-1}t \in V$. So, given $(y_1, y_2) \in U$ and $s, t \in G$ satisfying $s^{-1}t \in V$, then $(x_i, y_1) \in U$ for some i and $(x_i, y_2) \in U_1$, hence

$$|f(y_1, s) - f(y_2, t)| \leq |f(y_1, s) - f(x_i, s)| + |f(x_i, s) - f(x_i, t)| + |f(x_i, t) - f(y_2, t)| < \epsilon.$$

We are done.

REMARK. Since $C(X) = AP(X)$ for any topological left-zero semigroup X, the first example given in the remark following Theorem 1.1 shows that $AP \backslash (U_L \cup U_R)$ can be nonvoid if X is not compact.

The last positive result of this section is a theorem about semitopological left-groups whose underlying topological spaces are of metric type and is a generalization of a theorem of Rao [15, Theorem 2]; the main idea of the proof is taken from Rao's proof.

THEOREM 1.4. *Let $S = X \times G$ be a semitopological left-group where X is a first countable topological space and G is a complete metric space. Then $RUC = RMC$.*

PROOF. We assume S is not discrete and must show $RMC \subset RUC$. We do this by showing $f \notin RMC$ if $f \notin RUC$. Suppose $f \notin RUC$. Then there exist

$\epsilon > 0$, $\{(x_n, t'_n)\} \subset S$, $\{(y_n, s'_n)\} \subset S$ and $(y', s) \in S$ such that $\lim_n (y_n, s'_n) = (y', s)$ and

$$|f((x_n, t'_n)(y_n, s'_n)) - f((x_n, t'_n)(y, s))| = |f(x_n, t'_n s'_n) - f(x_n, t'_n s)| \geqslant \epsilon$$

for all n. We write $t'_n s'_n = t_n$, $t'_n s = s_n$, have

$$\lim_n s_n^{-1} t_n = e \in G,$$

and may assume:

(i) $\lim_n f(x_n, t_n) = 0$,

(ii) $\lim_n f(x_n, s_n) = 1$.

We now fix a point $y \in X$ and consider the sequence $\{L_{(x_n, s_n)} f\}$ of left translates of f. At least one of the following two situations must arise.

1. There is a sequence $\{(y, r_k)\} \subset S$ such that

$$\lim_k (y, r_k) = (y, e)$$

and a subsequence $\{L_{(x_m, s_m)} f\}$ of $\{L_{(x_n, s_n)} f\}$ such that

$$\lim_m L_{(x_m, s_m)} f(y, r_k) = p_k$$

with $|p_k| \leqslant \frac{1}{2}$ for each k. If μ is a cluster point in βS of the sequence of evaluation functionals on $C(S)$ corresponding to $\{(x_n, s_n)\}$, then

$$|\mu(R_{(y, r_k)} f)| = \lim_m |L_{(x_m, s_m)} f(y, r_k)| = |p_k| \leqslant \frac{1}{2}$$

for all k, while

$$\mu(R_{(y,e)} f) = \lim_m f(x_m, s_m) = 1.$$

Since $\lim_k (y, r_k) = (y, e)$, this implies $f \notin RMC$ [12, Theorem 3.1].

2. If d denotes the metric in G, there is a disc

$$D((y, e), \delta) = \{(y, u) \in S \,|\, d(u, e) < \delta\}$$

with $\delta > 0$ and a subsequence $\{L_{(x_m, s_m)} f\}$ of $\{L_{(x_n, s_n)} f\}$ such that

$$|L_{(x_m, s_m)} f(y, t)| \leqslant \frac{1}{2}$$

for only finitely many m for each $(y, t) \in D((y, e), \delta)$. We let ξ be a cluster point in βS of the sequence of evaluation functionals on $C(S)$ corresponding to $\{(x_m, t_m)\}$, hence

$$\xi(f) = \xi(R_{(y,e)}f) = 0,$$

and find a sequence $\{(y, r_k)\} \subset S$ such that $\lim_k (y, r_k) = (y, e)$, but $|\xi(R_{(y,r_k)}f)| \geq ½$ for all k. This will imply $f \notin RMC$.

For a fixed $k \geq 1$, we find (y, r_k) as follows. Choose $(y, u_k) \in S$ such that $u_k \neq e$ and $d(u_k, e) < 2^{-k}\delta$, let

$$D_k = D((y, u_k), d(u_k, e)/2)^- \quad \text{(closure in } S\text{)},$$

and let $E_m = \{(y, r) \in D_k \mid |L_{(x_m,s_m)}f(y, r)| \geq ½\}$. Since $D_k \subset D((y, e), \delta)$,

$$D_k = \bigcup_{i \geq 1} \left(\bigcap_{m > i} E_m \right).$$

The Baire category theorem applied to D_k asserts the existence of an i_0 such that $\bigcap_{m \geq i_0} E_m$ has nonvoid interior E^0 in D_k. Thus, there is a $(y, r_k) \in E^0$ and a $\delta_k > 0$ such that $D((y, r_k), \delta_k) \subset E^0$. Explicitly, for all $(y, r) \in D((y, r_k), \delta_k)$ and for all $m \geq i_0$

$$|L_{(x_m,s_m)}f(y, r)| \geq ½.$$

We now observe that, for a given $\epsilon > 0$,

$$|\xi(R_{(y,r_k)}f) - R_{(y,r_k)}f(x_m, t_m)| < \epsilon$$

for infinitely many m, that

$$(x_m, s_m)(y, s_m^{-1}) = (x_m, e)$$

for all m, that

$$\lim_m (y, s_m^{-1})(x_m, t_m)(y, r_k) = (y, r_k)$$

and hence that

$$|R_{(y,r_k)}f(x_m, t_m)| = |L_{(x_m,s_m)}((y, s_m^{-1})(x_m, t_m)(y, r_k))| \geq ½$$

for all large m, and have $|\xi(R_{(y,r_k)}f)| \geq ½$. The proof is complete.

The left-group $S = [0, 1] \times R$. [Here $[0, 1]$ and the real numbers R have their usual metric and topology. Also $U_L = U_R$ since R is abelian.] Consider the function $f \in C(S)$ defined as follows: for each natural number n, put

$$
(*) \qquad h_n(x) = \begin{cases} 2^{n+2}x - 2, & 2^{-n-1} \leqslant x \leqslant 3(2^{-n-2}), \\ -2^{n+2}x + 4, & 3(2^{-n-2}) \leqslant x \leqslant 2^{-n}, \\ 0, & \text{otherwise}, \end{cases}
$$

$$
f_n(s) = \begin{cases} (s-n)(n+1-s), & n \leqslant s \leqslant n+1, \\ 0, & \text{otherwise}, \end{cases}
$$

and define $f(x, s) = \Sigma_n h_n(x) f_n(s)$.

It is easy to check that $f \notin U_R = U_L = LUC$. However, since $|h_n(x)| \leqslant 1/4$ for all $x \in [0, 1]$ and for all n and the functions $\{f_n\}$ are uniformly equicontinuous on R [$\delta = \epsilon$ will do], Theorem 1.1, (iii) implies $f \in RUC$.

We next prove that $f \in WAP$. Let $\{(y_i, t_i)\}$ be a sequence in S; we must find a weakly convergent subsequence of $\{L_{(y_i,t_i)}f\}$. By assuming [as we may] that

$$\lim_i y_i = y_0 \in [0, 1]$$

and that $\lim_i t_i = t_0 \in R$ or $\lim_i |t_i| = \infty$, and dealing with some easy cases (using, where necessary the facts that $\{L_{(y_i,t_i)}f\} \subset C_{00}(S)$, the subset of $C(S)$ whose members have compact support, and that, in bounded subsets of $C_{00}(S)$, uniform convergence on compact subsets of S implies weak convergence), we are left with the only difficult case, the one where $y_0 = 0$ and $\lim_i |t_i| = \infty$.

In this case, for all large enough i we may assume

$$2^{-n_i-1} < y_i \leqslant 2^{-n_i}$$

for some natural number n_i. Also we may assume that the sequence

$$\{(2^{-n_i} - y_i)2^{n_i-1}\}$$

converges to y', $0 \leqslant y' \leqslant 1/4$, and that

$$\lim_i (n_i - t_i) = t' \in R \quad \text{or} \quad \lim_i |n_i - t_i| = \infty.$$

If $\lim_i |n_i - t_i| = \infty$, $\{L_{(y_i,t_i)}f\}$ converges to $0 \in C_{00}(S)$ uniformly on compact subsets of S, hence weakly. If $\lim_i (n_i - t_i) = t'$, then

$$\lim_i \|L_{(y_i,t_i)}f - g\| = 0,$$

where $g \in C(S)$ is defined by

$$
g(x, s) = \begin{cases} h_1(2^{-1} - y')(s - t')(t' + 1 - s), & t' \leqslant s \leqslant t' + 1, \\ 0, \quad \text{otherwise}. \end{cases}
$$

[h_1 is defined by (*) above.] Thus $f \in WAP$ as asserted.

Remarks about the function f. (a) The production of this function completes the proof of Theorem 1.1, (ii). It also completes the proof of Theorem 1.1, (iv), namely, it shows $RUC\backslash\psi^*(LUC)$ can be nonvoid. [To see the truth of the second assertion, one can use the fact that $\psi^*(f) \in WAP \subset RMC = RUC$, which follows from another part of Theorem 1.1, (iv) and Theorem 1.4.]

(b) For a locally compact left-group $S = X \times G$ with X compact, (A) of the Introduction holds as always. For the particular case $S = [0, 1] \times R$, we summarize the established breakdown of (B) of the Introduction.

$$WAP \not\subset LUC, \quad LUC \not\subset WAP, \quad LUC \neq LMC, \quad U_R \neq RUC.$$

However, we do still have $LUC = U_L$ [Theorem 1.1, (i)], $AP \subset U_L \cap U_R$ [Theorem 1.3], and $WAP \subset RUC$ [Theorem 1.4]. [The proof that $LUC \not\subset WAP$ can be achieved by extending a function $f \in LUC(R)\backslash WAP(R)$, such as $f(s) =$ arctan s, to a function F on S by means of the formula $F(x, s) = f(s)$. Obviously, such an F is in $LUC\backslash WAP = U_R\backslash WAP$.]

(c) In the next section the function f will be used to show that the weakly almost periodic compactification of $[0, 1] \times R$ is not the product of $[0, 1]$ and the weakly almost periodic compactification of R.

2. **Compactifications of products.** In [7, Corollary 4.4] deLeeuw and Glicksberg proved that the almost periodic compactification of a product of abelian topological semigroups with identity is the product of the almost periodic compactifications of the component semigroups. We show here that this conclusion remains true for a product of semitopological semigroups, one of which has a left identity, the other having a right identity. [Thus the result holds for left-groups.] We also give an example to illustrate the necessity of the hypotheses concerning the identities and then turn to the analogous assertion for the weakly almost periodic compactification which can be false even in familiar settings, but is true at least if one member of the product is a compact group. A positive result is established for one further compactification, that associated with the C^*-algebra RMC.

We first establish some notation. Let S be a semitopological semigroup. We regard the *almost periodic* [respectively, *weakly almost periodic*] *compactification aS* [*respectively, wS*] of S as a compact topological [respectively, semitopological] semigroup together with a continuous homomorphism α [respectively, ω] of S onto a dense subset of aS [respectively, wS], for which the following *universal mapping property* holds:

> if γ is a continuous homomorphism of S into a compact topological [respectively, semitopological] semigroup T, then there is a unique

continuous homomorphism γ_1 of aS [respectively, wS] into T such that

$$(\gamma_1 \circ \alpha)(s) = \gamma(s)$$

$$[\text{respectively, } (\gamma_1 \circ \omega)(s) = \gamma(s)]$$

for all $s \in S$.

A useful property of the adjoint map α^* [respectively, ω^*] is that it is an isometric isomorphism of $C(aS)$ [respectively, $C(wS)$] onto $AP(S)$ [respectively, $WAP(S)$]. See [3] for all these matters.

The next two lemmas are our basic techniques of proof in this section. They could be formulated in more general settings.

LEMMA 2.1. *Let S be a semitopological semigroup with subsets H and K such that $S = HK$ and let (δ, dS) be either (α, aS) or (ω, wS). Suppose the action*

$$(\delta H)^{-} \times (\delta K)^{-} \longrightarrow dS \quad [\textit{closure in } dS]$$

is jointly continuous. Then $dS = (\delta H)^{-}(\delta K)^{-}$.

PROOF. The desired conclusion is an immediate consequence of the facts that $(\delta H)^{-}(\delta K)^{-}$ contains δS which is dense in dS and that the continuous image of a compact set is compact.

LEMMA 2.2. *Suppose T_1 and T_2 are semitopological semigroups, T_1 having a right identity e_1 and T_2 having a left identity e_2. If $S = T_1 \times T_2$,* then

$$WAP(T_1) = WAP(S)|_{T_1 \times \{e_2\}}$$

and

$$WAP(T_2) = WAP(S)|_{\{e_1\} \times T_2}.$$

Similar statements hold for $AP(T_1)$ and $AP(T_2)$.

PROOF. The projections

$$(t_1, t_2) \longrightarrow (t_1, e_2)\colon\ S \longrightarrow T_1 \times \{e_2\},$$

$$(t_1, t_2) \longrightarrow (e_1, t_2)\colon\ S \longrightarrow \{e_1\} \times T_2$$

are retractions in the category of semitopological semigroups. The result follows from [2, Lemma 2].

THEOREM 2.3. *If $S = T_1 \times T_2$, where T_1 and T_2 are semitopological semigroups and T_1 has a right identity e_1 while T_2 has a left identity e_2, then $aS = aT_1 \times aT_2$.*

PROOF. We have $S = (T_1 \times \{e_2\})(\{e_1\} \times T_2)$. Since aS is always a *topological* semigroup, Lemma 2.1 implies

$$aS = \alpha(T_1 \times \{e_2\})^{-}\alpha(\{e_1\} \times T_2)^{-}.$$

[Here α is the continuous homomorphism of S into aS.] But we may identify $\alpha(T_1 \times \{e_2\})^{-}$ with aT_1 and $\alpha(\{e_1\} \times T_2)^{-}$ with aT_2, by Lemma 2.2. It remains to show that the equality $\tau_1\sigma_1 = \tau_2\sigma_2$ with $\tau_1, \tau_2 \in aT_1$, $\sigma_1, \sigma_2 \in aT_2$, implies $\tau_1 = \tau_2$ and $\sigma_1 = \sigma_2$. Suppose, for example, that $\tau_1 \neq \tau_2$. Then there is a function $f \in C(aT_1)$ with $f(\tau_1) = 1, f(\tau_2) = 0$. The image of f in $AP(T_1)$ under the adjoint of the map from T_1 into aT_1 can be extended to a function $g \in AP(S)$ which satisfies $g(t, s) = g(t, e_2)$ for all $t \in T_1$, $s \in T_2$. But then this g is the image under α^* of a function $h \in C(aS)$ which satisfies

$$h(\tau\sigma) = h(\tau) = f(\tau)$$

for all $\tau \in aT_1$, $\sigma \in aT_2$; in particular

$$1 = f(\tau_1) = h(\tau_1\sigma_1) = h(\tau_2\sigma_2) = f(\tau_2) = 0.$$

Thus $\tau_1 = \tau_2$. That $\sigma_1 = \sigma_2$ can be proved similarly.

Theorem 2.3 implies, in particular, that

$$a(X \times G) = aX \times aG = \beta X \times aG$$

for left-groups $X \times G$, a result that is implicit in [1].

An example is now given which shows the necessity of the hypotheses concerning the identities.

EXAMPLE. Let $[0, 1]$ and the real numbers R each be given the left-zero multiplication $x_1x_2 = x_1$. Then $S = [0, 1] \times R$ is also a left-zero semigroup and the function $f \in C(S)$ defined by $f(x, y) = \sin xy$ [where xy is the ordinary product of real numbers] is in $AP(S)$ since $AP(S) = C(S)$ for left-zero semigroups. The existence of such an $f \in AP(S)$ implies $aS \neq a[0, 1] \times aR$. [See Remark (iii) following Theorem 2.5 ahead.]

The only way in which the proof of the following theorem need differ from that of Theorem 2.3 is in the use of Ellis' theorem [9] to assert the joint continuity of the action $wS \times \omega(G) \longrightarrow wS$.

THEOREM 2.4. *If $S = T \times G$ where G is a compact topological group and T is a semitopological semigroup with right identity, then $wS = wT \times G$.*

The next theorem gives a criterion for determining when a compactification of a product semigroup is the product of the corresponding compactifications of the component semigroups. The theorem is a generalization of part of a theorem of Ptak [14, (6.2)] and could be used to prove Theorems 2.3 and 2.4; in fact, deLeeuw and Glicksberg proved Corollary 4.4 of [7] using some of the ideas involved in this theorem and I. Glicksberg has pointed out to us that much of Theorem 2.4 can be proved in this way as well. As with Lemmas 2.1 and 2.2, Theorem 2.5 could be formulated in a more general setting.

THEOREM 2.5. *Let $S = T_1 \times T_2$ be a product of semitopological semigroups where T_1 has a right identity e_1 and T_2 has a left identity e_2. For $f \in C(S)$, define $f_t \in C(T_2)$ and $f^s \in C(T_1)$ by*

$$f_t(s) = f^s(t) = f(t, s)$$

for $t \in T_1$, $s \in T_2$ and put

$$A_f = \{f_t \mid t \in T_1\}, \qquad B_f = \{f^s \mid s \in T_2\}.$$

Then $A_f \subset WAP(T_2)$ and $B_f \subset WAP(T_1)$ for every $f \in WAP(S)$. Also $wS = wT_1 \times wT_2$ if and only if one of A_f or B_f is relatively compact (in the norm topology) for every $f \in WAP(S)$. Thus both A_f and B_f are relatively compact for every $f \in WAP(S)$ if $wS = wT_1 \times wT_2$. Similar statements hold for the almost periodic compactification and the RMC-compactification (which is discussed later in this section).

PROOF. Let $f \in WAP(S)$. On account of the equalities

$$R_{s_1} f_t(s) = R_{(e_1, s_1)} f(t, s) \quad \text{and} \quad L_{t_1} f^s(t) = L_{(t_1, e_2)} f(t, s),$$

we see that the right orbit of f_t in $C(T_2)$ [respectively, left orbit of f^s in $C(T_1)$] is a subset of the restriction to $\{t\} \times T_2$ [respectively, to $T_1 \times \{s\}$] of the right orbit [respectively, left orbit] of f in $C(S)$. Since restriction maps are norm, hence weakly, continuous, $A_f \subset WAP(T_2)$ and $B_f \subset WAP(T_1)$.

Suppose $wS = wT_1 \times wT_2$. We wish to show, for example, that B_f is relatively compact in $WAP(T_1)$. [Note that this is the same as showing B_f is relatively compact in $C(T_1)$.] To do this, we put

$$B'_f = \{\hat{f}^\sigma \in C(wT_1) \mid \sigma \in wT_2\},$$

where $\hat{f} \in C(wT_1 \times wT_2)$ satisfies $\omega^* \hat{f} = f$ and $\hat{f}^\sigma(\tau) = \hat{f}(\tau, \sigma)$ for $\tau \in wT_1$, $\sigma \in wT_2$. B'_f is compact, being the image under the continuous function $\sigma \longrightarrow \hat{f}^\sigma$ of the compact set wT_2. The continuity of this function follows from an elementary property of the product topology and the compactness of wT_1. Since $B_f \subset \omega^* B'_f$, we have reached the desired conclusion.

Conversely, suppose, for example, that B_f is relatively compact for a given $f \in WAP(S)$. Suppose, as well, that $\tau \in wT_1$ and $\sigma \in wT_2$, and $\{t_\gamma\} \subset T_1$ and

$\{s_\delta\} \subset T_2$ are nets such that

$$\lim_\gamma g(t_\gamma) = \hat{g}(\tau) \quad \text{and} \quad \lim_\delta h(s_\delta) = \hat{h}(\sigma)$$

for all $g \in WAP(T_1)$ and $h \in WAP(T_2)$, respectively. [Here, for example, $\hat{g} \in C(wT_1)$ is such that $\omega^*\hat{g} = g$.] The relative compactness of B_f tells us that a subnet of $\{f^{s_\delta}\}$ converges uniformly on T_1 to a function $g \in WAP(T_1)$. Since $f_t \in WAP(T_2)$ for all $t \in T_1$,

$$\lim_\delta f^{s_\delta}(t) = \lim_\delta f_t(s_\delta) = (f_t)\hat{\ }(\sigma),$$

hence the function g is uniquely determined and

$$\lim_\delta \|f^{s_\delta} - g\| = 0.$$

Since $g \in WAP(T_1)$, we have the joint limit

$$\lim_{\gamma,\delta} |f(t_\gamma, s_\delta) - g(\tau)| \leqslant \lim_{\gamma,\delta}(|f^{s_\delta}(t_\gamma) - g(t_\gamma)| + |g(t_\gamma) - \hat{g}(\tau)|) = 0.$$

Hence, if κ: $C(wT_1 \times wT_2) \longrightarrow C(S)$ is defined by

$$\kappa F(t, s) = F(\omega t, \omega s),$$

it follows that $\kappa(C(wT_1 \times wT_2)) \supset WAP(S)$; and it is easy to see the reverse inclusion. So $wS = wT_1 \times wT_2$.

REMARKS. (i) The existence of the weakly almost periodic function f that was constructed on the left-group $[0, 1] \times R$ in §1 implies that $w([0, 1] \times R) \neq [0, 1] \times wR$ since $B_f \subset C([0, 1])$ is not equicontinuous and hence is not relatively compact.

(ii) In [7] it is asserted that $w(T_1 \times T_2)$ is not always equal to $wT_1 \times wT_2$, where T_1 and T_2 are abelian topological semigroups with identity. I. Glicksberg has informed us that the example that was intended is $R \times R$. The product of the integers $Z \times Z$ is another example and can be quickly handled. Using Grothendieck's criterion [10, Proposition 7], one can easily see that the function

$$f(m, n) = \begin{cases} 1, & \text{if } m = n, \\ 0, & \text{if } m \neq n, \end{cases}$$

is in $WAP(Z \times Z)$, while B_f is not relatively compact. Thus $w(Z \times Z) \neq wZ \times wZ$.

(iii) The function f defined on the left-zero semigroup $[0, 1] \times R$ following

Theorem 2.3 clearly has B_f not relatively compact and, although this product semigroup does not satisfy the hypotheses concerning the identities of Theorem 2.5, one can still use the argument of the second paragraph of the proof of that theorem, arguing by contradiction, to conclude that

$$a([0, 1] \times R) \neq [0, 1] \times aR.$$

(iv) The first conclusion of Theorem 2.5 can fail if one of the component semigroups is allowed to have no identity of any kind. For example, if $S = [0, 1] \times R$ has the multiplication

$$(x, s)(y, t) = (0, s + t),$$

then the function f defined on S by $f(x, s) = \sin xs$ is in $AP(S)$ while neither A_f nor B_f is relatively compact.

Our methods in this section can be applied to get a positive result about another compactification of a semitopological semigroup S, that corresponding to $RMC(S)$. We regard this compactification mS as a compact semigroup together with a continuous homomorphism μ of S onto a dense subset of mS: in mS the maps

$$(*) \qquad x \longrightarrow yx \quad \text{and} \quad x \longrightarrow x\mu(s)$$

from mS are continuous at least for all $y \in mS$ and for all $s \in S$ and a universal mapping property is satisfied [12, Theorem 3.4]. Namely, if γ is a continuous homomorphism of S into a compact semigroup T with continuity properties analogous to $(*)$, then there is a unique continuous homomorphism γ_1 of mS into T such that

$$(\gamma_1 \circ \mu)(s) = \gamma(s)$$

for all s in S.

THEOREM 2.6. *If $S = T \times G$ where G is a compact topological group and T is a semitopological semigroup with right identity, then $mS = mT \times G$.*

PROOF. The conclusion of Lemma 2.1 also holds if $(\delta, dS) = (\mu, mS)$ and we can use Ellis' theorem, as in the proof of Theorem 2.4, to assert the joint continuity of the action

$$mS \times \mu(G) \longrightarrow mS.$$

The rest of the proof is like that of Theorem 2.3.

REMARKS. (i) The existence of the function in $LUC \backslash RUC \subset LMC$ on the left-group $R \times [0, 1)$ constructed in the Remark following Theorem 1.1 shows that an *LMC* analogue of Theorem 2.6 does not exist. It also shows that the hypothesis that T have a right identity is essential for the conclusion of Theorem 2.6, i.e., if S is the topological *right-group*

$$R \times [0, 1) \quad [(x, s)(y, t) = (y, (s + t)(\text{mod } 1))],$$

then $mS \neq mR \times m[0, 1) = mR \times [0, 1)$.

(ii) In the special case that the extra hypotheses that T is a left-group and T and G are first countable are made, the conclusion of Theorem 2.6 can also be proved using Corollary 1.2 and Theorems 1.4 and 2.3. In this case we have

$$mS = mT \times G = aT \times G.$$

3. **Analytic weakly almost periodic functions.** Certain classical questions about almost periodic functions of a real variable and analytic almost periodic functions are considered in this section. An "analytic almost periodic" function is not defined to be an entire function which is almost periodic on the topological group C for the obvious reason that the only such functions are constants. Bohr defined analytic almost periodic functions to be almost periodic functions depending on a real parameter which are analytic on a strip (or half-plane). We are led to an obvious generalization. For notation, let

$$-\infty \leqslant a < c \leqslant d < b \leqslant +\infty,$$

and let

$$(a, b) = \{z \in C: a < \operatorname{Re} z < b\}$$

and

$$[c, d] = \{z \in C: c \leqslant \operatorname{Re} z \leqslant d\}.$$

Let $A(a, b)$ be the space of analytic functions on (a, b), and let $A^*(a, b)$ denote the subspace consisting of those functions $f \in A(a, b)$ with $f|_{[c,d]}$ bounded for every closed strip $[c, d] \subset (a, b)$. Consider $[c, d] \subset R^2$, and let $S_{[c,d]}$ denote the locally compact topological left-group $X \times G$, where X is the real interval $[c, d]$ and G is the additive group of real numbers.

DEFINITION. Let $f \in A^*(a, b)$. If $f \in WAP(S_{[c,d]})$ [respectively, $f \in AP(S_{[c,d]})$] for every closed strip $[c, d] \subset (a, b)$ then f is called *analytic weakly almost periodic* [respectively, *analytic almost periodic*] on (a, b).

Parallel definitions for the spaces *LMC*, *LUC*, *RMC*, *RUC*, and $U_L = U_R$ are possible, of course, but the next theorem shows that we would get $A^*(a, b)$ in each such case. The example following this theorem shows that the subspace of analytic weakly almost periodic functions is proper in $A^*(a, b)$.

THEOREM 3.1. *Let $f \in A^*(a, b)$. Then f and all its derivatives are uniformly continuous on every closed strip $[c, d] \subset (a, b)$.*

PROOF [5, Theorem 3.7, p. 72].

EXAMPLE. The function f defined on the strip (0, 1) by

$$f(z) = \sum_{n=1}^{\infty} (ni - z)^{-2}$$

is a member of $A^*(a, b)$, but $g \in C(R)$ defined by

$$g(s) = f(x_0 + is)$$

is not in $WAP(R)$ for any $x_0 \in (0, 1)$. Thus f is not analytic weakly almost periodic. (See Theorem 3.5 ahead in this regard.)

We will consider analogues of the following well-known classical theorems:

A. *The derivative of an almost periodic function of a real variable is almost periodic provided that it is uniformly continuous.*

B. *An antiderivative of an almost periodic function of a real variable is almost periodic provided that it is bounded.*

C. *If the restriction of $f \in A^*(a, b)$ to any single vertical line is an almost periodic function of a real variable, then f and all its derivatives are analytic almost periodic on (a, b).*

Theorem B is the basis for the vast literature on almost periodic solutions to ordinary differential equations. We show by example (following Theorem 3.3) that, unfortunately, the analogue for weakly almost periodic functions of Theorem B is not true; and, therefore, the development of a theory of weakly almost periodic solutions to differential equations cannot be so fruitful. The analogues of A and C are given in Theorems 3.3 and 3.5 and Corollary 3.7 below.

LEMMA 3.2. *Let G be a locally compact group. If $f \in WAP(G)$ and $g \in L^1(G)$, then the convolution $f * g$ is weakly almost periodic.*

PROOF. Let μ be left Haar measure on G. Note that if λ is a continuous linear functional on $WAP(G)$, then the function

$$s \longrightarrow \lambda(L_s f)$$

is weakly almost periodic [6, Lemma 5.13]. Whence, since

$$f * g(s) = \int L_s f(t) g(t^{-1}) \, d\mu(t),$$

we have that $f * g \in WAP(G)$.

THEOREM 3.3. *Suppose that $f \in WAP(R)$ and that f is everywhere differentiable. Then the derivative f' is weakly almost periodic if and only if f' is uniformly continuous.*

PROOF. That weakly almost periodic functions on locally compact groups are uniformly continuous is well known. Suppose, now, that f' is uniformly continuous. Then, given $\epsilon > 0$, we can choose a continuously differentiable function ψ with compact support so that

$$\|f' * \psi - f'\|_\infty < \epsilon.$$

Integrating by parts and using the fact that ψ has compact support, we get that

$$\begin{aligned} f' * \psi(x) &= \int f'(x-y)\psi(y)\,dy \\ &= \int f(x-y)\psi'(y)\,dy = f * \psi'(x). \end{aligned}$$

But since ψ has compact support, so does ψ'. Thus, by Lemma 3.2, we have that $f' * \psi$ is weakly almost periodic. But then, f' may be uniformly approximated by weakly almost periodic functions; since $WAP(R)$ is norm closed in $C(R)$, we conclude that $f' \in WAP(R)$.

EXAMPLE. Let f be the inverse trigonometric function

$$f(x) = \arctan x.$$

Then, since the derivative f' vanishes at infinity, the derivative of f is weakly almost periodic. However, f is not weakly almost periodic even though it is bounded (cf., statement B). To see that f is not weakly almost periodic, take any sequence $n \longrightarrow x_n$ in R which goes to $+\infty$; take, say, $x_n = n$. Then, since $\{L_n(f)\}$ converges pointwise on R to the constant function $\pi/2$, we have that $\pi/2$ is the only possible weak cluster point of $\{L_n(f)\}$. However, weak convergence on R is the same as pointwise convergence on the Stone-Čech compactification of R, and the constant function $\pi/2$ is not a pointwise cluster point of the sequence of extended functions.

For Theorem 3.5, we will need the following corollary to the Vitali Convergence Theorem:

THEOREM 3.4. *Let $\{f_n\}$ be a uniformly bounded sequence of functions in $A(c, d)$. Then there is a subsequence of $\{f_n\}$ which converges uniformly on compact subsets to a limit F; the limit being, therefore, analytic in (c, d).*

PROOF [16, Corollary 5.22, p. 169].

THEOREM 3.5. *Let $f \in A^*(a, b)$. Suppose that, for some x_0 with $a < x_0 < b$, the function g defined by*

$$g(t) = f(x_0 + it), \qquad t \in R,$$

is in $WAP(R)$. Then f is analytic weakly almost periodic on (a, b).

PROOF. Let $\{a_n\}$ be a sequence of real numbers, and let $z_n = x_0 + ia_n$. Considering the strip (a, b) to be a left-group, then

$$R_{z_n} f(s + it) = f(s + i(t + a_n)).$$

By Theorem 3.4 we may assume, by passing to a subsequence if necessary, that the sequence of analytic functions $\{R_{z_n} f\}$ converges uniformly on compact subsets to a function $F \in A^*(a, b)$. Define a function G on R by

$$G(t) = F(x_0 + it).$$

Since $R_{a_n} g = R_{z_n} f|_{x_0 + iR}$, we have that G is the only pointwise cluster point of the sequence $\{R_{a_n} g\}$. Consequently, since $g \in WAP(R)$, we must have that $\{R_{a_n} g\}$ converges weakly to G.

We now want to show that $\{R_{z_n} f\}$ converges weakly to F on every closed strip $[c, d] \subset (a, b)$, with $c \leqslant x_0 \leqslant d$. It suffices to show that $\{R_{z_n} f\}$ converges quasi-uniformly to F, since a sequence of bounded continuous functions $\{g_n\}$ converges weakly to a continuous function G if and only if it is uniformly bounded and, together with every subsequence, converges to G quasi-uniformly [8, Lemma 30, p. 281]. (A sequence $\{g_n\}$ converges *quasi-uniformly* to a function G if it converges pointwise to G and if, given $\epsilon > 0$ and index N, there are indices $n_k \geqslant N$, $k = 1, 2, \ldots, m$, such that

$$\min_{1 \leqslant k \leqslant m} |g_{n_k}(t) - G(t)| < \epsilon$$

for every t.)

Let $\epsilon > 0$. Since f and F are both uniformly continuous on $[c, d]$, there is a $\delta > 0$ such that if $|x_1 - x_2| < \delta$, then

$$|f(x_1 + it) - f(x_2 + it)| < \epsilon/3$$

and

$$|F(x_1 + it) - F(x_2 + it)| < \epsilon/3$$

for every $t \in R$. Also, given an index N, there are indices $n_k \geqslant N$, $k = 1, 2, \ldots, m$, such that

$$\min_{1 \leqslant k \leqslant m} |g(t + a_{n_k}) - G(t)| < \epsilon/3$$

for every $t \in R$; which is to say,

$$\min_{1 \leqslant k \leqslant m} |f(x_0 + i(t + a_{n_k})) - F(x_0 + it)| < \epsilon/3$$

for every $t \in R$. Thus, for any x with $c \leqslant x \leqslant d$ and $|x - x_0| < \delta$, we get

$$\begin{aligned} &|f(x + i(t + a_{n_k})) - F(x + it)| \\ &\qquad \leqslant |f(x + i(t + a_{n_k})) - f(x_0 + i(t + a_{n_k}))| \\ &\qquad\quad + |f(x_0 + i(t + a_{n_k})) - F(x_0 + it)| \\ &\qquad\quad + |F(x_0 + it) - F(x + it)|, \end{aligned}$$

and the least of these is less than ϵ. Whence, $\{R_{z_n} f\}$ converges quasi-uniformly to F on the strip $(x_0 - \delta, x_0 + \delta) \cap [c, d]$. Since δ did not depend on x_0, we may choose another point and repeat; we conclude that $\{R_{z_n} f\}$ converges quasi-uniformly to F on all of the strip $[c, d]$, as desired.

COROLLARY 3.6 (CF. [5, THEOREM 3.12]). *If $\{f_n\}$ is a sequence of analytic weakly almost periodic functions on a strip (a, b) which converges uniformly on every closed strip $[c, d] \subset (a, b)$ to a function f, then f is analytic weakly almost periodic on (a, b).*

COROLLARY 3.7. *If the function f is analytic weakly almost periodic on the strip (a, b), then all its derivatives are analytic weakly almost periodic in (a, b).*

PROOF. 3.1, 3.3, and 3.5.

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